

More about Birkhoff's Invariant and Thorne's Hoop Conjecture for Horizons

M. Cvetič^{*}, G.W. Gibbons[†] and C.N. Pope^{‡,†}

^{}Department of Physics and Astronomy,
University of Pennsylvania, Philadelphia, PA 19104, USA*

^{}Center for Applied Mathematics and Theoretical Physics,
University of Maribor, Maribor, Slovenia*

*[†]DAMTP, Centre for Mathematical Sciences,
Cambridge University, Wilberforce Road, Cambridge CB3 0WA, UK*

*[‡]George P. & Cynthia W. Mitchell Institute for
Fundamental Physics and Astronomy,
Texas A&M University, College Station, TX 77843-4242, USA*

Abstract

A recent precise formulation of the hoop conjecture in four spacetime dimensions is that the Birkhoff invariant β (the least maximal length of any sweepout or foliation by circles) of an apparent horizon of energy E and area A should satisfy $\beta \leq 4\pi E$. This conjecture together with the Cosmic Censorship or Isoperimetric inequality implies that the length ℓ of the shortest non-trivial closed geodesic satisfies $\ell^2 \leq \pi A$. We have tested these conjectures on the horizons of all four-charged rotating black hole solutions of ungauged supergravity theories and find that they always hold. They continue to hold in the presence of a negative cosmological constant, and for multi-charged rotating solutions in gauged supergravity. Surprisingly, they also hold for the Ernst-Wild static black holes immersed in a magnetic field, which are asymptotic to the Melvin solution. In five spacetime dimensions we define β as the least maximal area of all sweepouts of the horizon by two-dimensional tori, and find in all cases examined that $\beta(g) \leq \frac{16\pi}{3}E$, which we conjecture holds quite generally for apparent horizons. In even spacetime dimensions $D = 2N + 2$, we find that for sweepouts by the product $S^1 \times S^{D-4}$, β is bounded from above by a certain dimension-dependent multiple of the energy E . We also find that ℓ^{D-2} is bounded from above by a certain dimension-dependent multiple of the horizon area A . Finally, we show that ℓ^{D-3} is bounded from above by a certain dimension-dependent multiple of the energy, for all Kerr-AdS black holes.

Contents

1	Introduction	2
2	Birkhoff Bound in Four Spacetime Dimensions	6
2.1	Four-dimensional asymptotically-flat black holes	6
2.2	Four-dimensional asymptotically-AdS black holes	7
2.3	Four-dimensional asymptotically-Melvin black holes	8
3	Birkhoff Bound in Five Spacetime Dimensions	9
3.1	Five-dimensional asymptotically-flat black holes	10
3.2	Five-dimensional asymptotically-AdS black holes	11
4	Closed Geodesic Bounds	13
4.1	Asymptotically-AdS black holes in higher dimensions	13
4.1.1	$D = 2N + 2$ dimensions	15
4.1.2	$D = 2N + 1$ dimensions	17
5	Sweepouts by Higher-Dimensional Spheres	17
5.1	$S^1 \times S^{D-4}$ sweepouts in $D = 2N + 1$ Kerr-AdS	18
5.2	Sweepouts of S^{D-2} by $S^p \times S^q$	19
5.2.1	Higher-dimensional magnetic fields	19
5.2.2	Non-rotating Einstein-Maxwell-Dilaton black holes in higher dimensions . . .	20
5.2.3	Kerr-AdS with a single rotation parameter	20
5.2.4	Myers-Perry with an applied magnetic field along one non-rotating direction	20
6	Conclusions	21
7	Acknowledgments	21
A	Isometric embedding of the Ernst-Wild horizon in \mathbb{E}^3	21

1 Introduction

Many years ago Thorne conjectured [1] that in four spacetime dimensions

Horizons form when and only when a mass E gets compacted into a region whose circumference in EVERY direction is $C \leq 4\pi E$.

Since that time there has been a great deal of work making the idea more precise, and attempting to establish its correctness or otherwise (see e.g. [2, 3, 4]).¹ Since then the hoop conjecture has been invoked in numerical relativity (see e.g. [5, 6]) and studies of hole scattering in four and higher dimensions [7, 8, 9, 10, 11, 12]. It has also been suggested that the hoop conjecture may provide a route to a precise formulation of the idea that there is a minimal length in quantum gravity [13].

¹Indeed, as we discuss in the appendix, one interpretation of Thorne’s original statement of the conjecture appears to be violated by black holes in external magnetic fields.

To begin with one needs a definition of the total energy E . One obvious possibility, in the asymptotically flat case that Thorne had in mind, is to take the ADM mass. In order to define the circumference one needs a notion of a surface that surrounds the matter. Thus one is led to consider a Cauchy surface Σ containing an outermost marginally trapped surface or “apparent horizon” S with induced metric g , and to assign to the pair $\{S, g\}$ a hoop radius R or circumference $C = 2\pi R$. In a recent note [14] it has been suggested that for topologically-spherical apparent horizons with metric g in four-dimensional spacetimes one may take for C the Birkhoff invariant $\beta(g)$, and so we propose²

Conjecture 1: *The Birkhoff invariant $\beta(g)$ and the energy E of an apparent horizon, in $3 + 1$ dimensions, satisfy*

$$\boxed{\beta(g) \leq 4\pi E.} \quad (1.1)$$

Some evidence for conjecture 1 was presented in [14].³ In the appendix we shall give some further discussion on the appropriateness of taking $\beta(g)$ as the definition of the circumference, or hoop radius. The bulk of our paper is concerned with whether or not conjecture 1, and related inequalities in four and higher dimensions, are valid.

The Birkhoff invariant is defined as follows. If the matter obeys the dominant energy condition (which it does for ungauged supergravity), we can assume in four spacetime dimensions that the apparent horizon is topologically spherical [15, 16, 17, 18]. Now, suppose that $S = \{S^2, g\}$ is a sphere with arbitrary metric g and $f : S \rightarrow \mathbb{R}$ is a function on S with just two critical points, a maximum and a minimum. Each level set $f^{-1}(c)$, $c \in \mathbb{R}$, has a length $\ell(c)$, and for any given function f we may define

$$\beta(g; f) = \max_c \ell(c). \quad (1.2)$$

(For example, for the ordinary unit sphere with spherical polar coordinates (θ, ϕ) , we may take $f = \cos \theta$ and $\ell(\cos \theta) = 2\pi \sin \theta$. Thus $\beta(g; \cos \theta) = 2\pi$.) We now define the Birkhoff invariant $\beta(S, g)$ by minimizing $\beta(g; f)$ over all such functions,

$$\beta(g) = \inf_f \beta(g; f). \quad (1.3)$$

The intuitive meaning of $\beta(g)$ is the least length of a closed flexible hoop that may be slipped over the surface S . To understand why, note that each function f gives a foliation of S by a one-parameter family of simple closed curves $f = c$ which we may think of as the hoop at each “moment of time” c . $\beta(g; f)$ is the greatest length of the hoop during this process. If we change the foliation we can hope to reduce this greatest length, and the infimum is the best that we can do. The phrase “moment of time” is in quotation marks because we are not regarding f as a physical time function, but merely as a convenient way of thinking about the geometry of S .

Clearly the definition of $\beta(g)$ does not depend upon the spacetime’s being asymptotically flat. Thus one is led to conjecture that it continues to hold for asymptotically-AdS spacetimes, with the ADM mass being replaced by the Abbott-Deser mass. Another possibility is to consider a static

²For this and all subsequent conjectures, we assume that the dominant energy condition holds.

³For the purposes of the present work we are only interested in the necessity of this proposed inequality, and moreover we shall not discuss to what extent it captures all of what Thorne had in mind when he made his original conjecture. We shall also not be concerned with the question of whether the ADM mass may be replaced by some quasi-local notion of mass.

black hole immersed in an asymptotically Melvin magnetic field, for which an appropriate notion of total energy is available. In section 2 of this paper we shall confirm this conjecture for all the exact stationary black hole solutions known to us. Note that to confirm the conjecture it suffices to bound $\beta(g; f)$ from above by $4\pi E$ for some particular, conveniently chosen, foliation f . We do not need to calculate $\beta(g)$ itself.

It was shown by Birkhoff [19] that there is at least one closed geodesic γ on S with length $\ell(\gamma) = \beta(g)$. It follows that if $\ell(g)$ is the length of the shortest non-trivial closed geodesic on S , then

$$\ell(g) \leq \beta(g) \tag{1.4}$$

and so, if our conjecture is correct, it should be the case that

Conjecture 2: *The length $\ell(g)$ of the shortest geodesic and the energy E of an apparent horizon, in $3 + 1$ dimensions, satisfy*

$$\boxed{\ell(g) \leq 4\pi E.} \tag{1.5}$$

Again, to confirm conjecture 2 it suffices to bound $\ell(\gamma)$ by $4\pi E$ for some particular, conveniently chosen, geodesic γ . We do not need to calculate $\ell(g)$ itself. The simplest case in which this can be done is if $\{S, g\}$ admits a fixed-point free isometric action of \mathbb{Z}_2 , an “antipodal map.” One may then pass to $S/\mathbb{Z}_2 \equiv \mathbb{RP}^2$. Since $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$, there must be at least one closed geodesic γ in this homotopy class, which may be obtained by minimizing the length amongst all non-trivial curves in this class. To obtain an upper bound for $\ell(\gamma)$, it suffices to find an upper bound for the distance between a point and its antipode.

In fact Pu [20] has shown in this case that if $A(g)$ is the area of S then

$$\boxed{\ell(g) \leq \sqrt{\pi A(g)}.} \tag{1.6}$$

However, the Penrose inequality [21] states that

$$\sqrt{\pi A} \leq 4\pi E, \tag{1.7}$$

and so the conjecture (1.5) holds for those apparent horizons $\{S, g\}$ admitting an antipodal map [14]. In fact all event horizons of regular black holes solutions known to us in four spacetime dimensions admit an antipodal map and thus satisfy (1.5).

Given the current interest in higher dimensions, it is natural to attempt to extend these conjectures beyond four dimensions, and then to test them against known exact solutions. This we do in section 3 of the present paper, for most of the exact five-dimensional black holes solutions known to us that have as horizon a topological 3-sphere. In general, these have $\{S, g\} \equiv \{S^3, g\}$ for which g is not the round 3-sphere metric. In the cases that we study it is invariant under the action of $U(1) \times U(1)$. Thus Birkhoff’s invariant is obtained by considering the area of the leaves of a Clifford type foliation of S^3 by 2-tori $S^1 \times S^1$ with two singular linked S^1 leaves. For a review of mathematical results on such higher-dimensional “sweepouts,” the reader may consult [22]. We propose

Conjecture 3: *The Birkhoff invariant $\beta(g)$ for $S^1 \times S^1$ sweepouts, and the energy E of an apparent horizon, in $4 + 1$ dimensions, satisfy*

$$\boxed{\beta(g) \leq \frac{16\pi}{3}\pi E.} \tag{1.8}$$

We find that this is satisfied in all the cases we have tested.

Based on an investigation of various higher-dimensional black hole examples, we find that an analogue of conjecture 2 in (1.5) holds in all cases. Thus we propose

Conjecture 4: *The length $\ell(g)$ of the shortest closed geodesic, and the energy E of an apparent horizon, in D spacetime dimensions, satisfy*

$$\ell(g)^{D-3} \leq \frac{32\pi^{D-2}E}{(D-2)\mathcal{A}_{D-2}}, \quad (1.9)$$

where \mathcal{A}_{D-2} is the volume of the standard round $(D-2)$ -sphere of unit radius. Note that in five dimensions, conjecture 4 does not follow from conjecture 3.⁴

The results in four dimensions described earlier strongly suggest that equation (1.6) holds for all apparent horizons, with or without an antipodal symmetry. There is no analogue of Pu's theorem in higher dimensions. Nevertheless our calculations suggest the validity of

Conjecture 5: *The length $\ell(g)$ of the shortest closed geodesic, and the $(D-2)$ -volume A of an apparent horizon, in D spacetime dimensions, satisfy, at least in even dimensions,*

$$\left(\frac{\ell(g)}{2\pi}\right)^{D-2} \leq \frac{A}{\mathcal{A}_{D-2}}. \quad (1.11)$$

We have verified that conjectures 4 and 5 are both satisfied for Kerr-AdS black holes in all even dimensions. We also find that conjecture 4 is satisfied for Kerr-AdS black holes in all odd dimensions. We have so far been unable to find a suitable bound for $\ell(g)^{D-2}/A$ that would support conjecture 5 in odd dimensions.

Note that in five spacetime dimensions the Penrose or Isoperimetric Inequality for black holes is [17, 23]

$$A \leq 2\pi^2 \left(\frac{8E}{3\pi}\right)^{\frac{3}{2}}, \quad (1.12)$$

but since, even if the metric admits an antipodal map, there appears to be no useful general inequality for $\frac{\ell^3(g)}{A(g)}$ [24, 25], this does not give us useful information about conjecture 4.

In D spacetime dimensions the known exact black hole solutions admit foliations of the S^{D-2} horizons by $T^{[\frac{D-1}{2}]}$ which have co-dimension larger than one if $D \geq 6$. Thus they cannot be used as “hyperhoops.” However, in the case of rotating black holes with a single non-vanishing rotation parameter we are able to construct a foliation of the horizon by leaves with topology $S^1 \times S^{D-4}$. This allows us to define the Birkhoff invariant in terms of the $(D-3)$ -volume of these “hyperhoops,” suggesting

Conjecture 6: *For sweepouts by $S^1 \times S^{D-4}$ hyperhoops, the Birkhoff invariant $\beta(g)$ and the energy E of an apparent horizon, in dimensions $D = 2N + 1$, satisfy*

$$\beta(g) \leq \frac{32\pi}{(2N-1)} (N-1)^{\frac{1}{2}(N+1)} N^{-\frac{1}{2}N} E. \quad (1.13)$$

⁴In fact for any metric g on T^2 Loewner has shown that for the shortest non-null homotopic curve

$$\ell(T^2, g) \leq \frac{2^{\frac{1}{2}}}{3^{\frac{1}{4}}} \sqrt{A(T^2, g)}. \quad (1.10)$$

We have verified this conjecture for all odd-dimensional Kerr-AdS black holes with a single non-vanishing rotation parameter, with or without an external magnetic field.

2 Birkhoff Bound in Four Spacetime Dimensions

In this section we shall test conjecture 1, given in (1.1), for three explicitly-known classes of black holes in four dimensions. To begin with, we consider the general 4-charged rotating black holes of $\mathcal{N} = 8$ ungauged supergravity. These are asymptotically flat, and generalise the examples considered in [14], which were restricted to the case of pairwise-equal charges. The second class we shall consider is rotating asymptotically-AdS black holes. These are solutions of $\mathcal{N} = 8$ gauged supergravity, with pairwise-equal charges. Finally, we shall consider solutions of Einstein-Maxwell theory in which a neutral black hole is immersed in a Melvin-type magnetic field. In all cases, we find an upper bound for the Birkhoff invariant $\beta(g)$, which is at most equal to the upper bound given by conjecture 1.

2.1 Four-dimensional asymptotically-flat black holes

These 4-charge solutions in ungauged $\mathcal{N} = 8$ supergravity were obtained in [26], and a convenient expression for them can be found in [27]:

$$\begin{aligned}
ds_4^2 &= -\frac{\rho^2 - 2mr}{W} (dt + Bd\phi)^2 + W \left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\rho^2 - 2mr} d\phi^2 \right), \\
B &= \frac{2ma(rc_{1234} - (r - 2m)s_{1234}) \sin^2 \theta}{\rho^2 - 2mr}, \\
W^2 &= r_1 r_2 r_3 r_4 + a^4 \cos^4 \theta + \\
&\quad a^2 [2r^2 + 2mr \sum_i s_i^2 + 8m^2 c_{1234} s_{1234} - 4m^2 (s_{123}^2 + s_{124}^2 + s_{134}^2 + s_{234}^2 + 2s_{1234}^2)] \cos^2 \theta, \\
\Delta &= r^2 - 2mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \\
r_i &= r + 2ms_i^2, \quad s_{i_1 \dots i_n} = s_{i_1} \dots s_{i_n}, \quad c_{i_1 \dots i_n} = c_{i_1} \dots c_{i_n},
\end{aligned} \tag{2.1}$$

where here, and throughout the paper, we use the abbreviations

$$s_i = \sinh \delta_i, \quad c_i = \cosh \delta_i. \tag{2.2}$$

The metric (2.1) depends on the mass parameter m , the rotation parameter a , and the four charge “boost” parameters δ_i . In order to avoid the unnecessary manipulation of square roots, it is convenient to use r_+ , the radius of the outer horizon, rather than m , in the parametrization. Thus we have $m = (r_+^2 + a^2)/(2r_+)$. Special cases include the Kerr metric when $s_i = 0$; the Schwarzschild metric if additionally $a = 0$; the Kerr-Newman metric if $s_i = s$; and the Reissner-Nordström metric if additionally $a = 0$.

We shall bound the Birkhoff invariant $\beta(g)$ from above by considering a foliation by circles $\theta = \text{constant}$. It is easily seen that on the horizon, $g_{\phi\phi}$ attains its maximum value at $\theta = \frac{1}{2}\pi$, and so setting $f = \theta$ in (1.2) we have

$$\beta(g) \leq \beta(g; \theta) = \frac{2\pi (r_+^2 + a^2) \left(r_+^2 \prod_i c_i + a^2 \prod_i s_i \right)}{r_+ \prod_j [r_+^2 c_j^2 + a^2 s_j^2]^{1/4}}. \tag{2.3}$$

The mass of the black hole is given by [26]

$$E = \frac{1}{4}m \sum_i (c_i^2 + s_i^2), \quad (2.4)$$

and therefore conjecture 1 will be satisfied if

$$\sum_i (c_i^2 + s_i^2) - \frac{4[\prod_i c_i + \tilde{a}^2 \prod_i s_i]}{\prod_j [c_j^2 + \tilde{a}^2 s_j^2]^{1/4}} \geq 0, \quad (2.5)$$

where we have defined $\tilde{a} \equiv a/r_+$. We can easily establish that (2.5) is satisfied, by observing that

$$\sum_i c_i^2 - \frac{4 \prod_i c_i}{\prod_j [c_j^2 + \tilde{a}^2 s_j^2]^{1/4}} \geq \sum_i c_i^2 - \frac{4 \prod_i c_i}{\prod_j c_j^{1/2}} = \sum_i c_i^2 - 4 \left(\prod_i c_i \right)^{1/2} \geq 0, \quad (2.6)$$

and that

$$\sum_i s_i^2 - \frac{4 \tilde{a}^2 \prod_i s_i}{\prod_j [c_j^2 + \tilde{a}^2 s_j^2]^{1/4}} \geq \sum_i s_i^2 - \frac{4 \tilde{a}^2 \prod_i s_i}{\prod_j [\tilde{a}^2 s_j^2]^{1/4}} = \sum_i s_i^2 - 4 \left(\prod_i s_i \right)^{1/2} \geq 0, \quad (2.7)$$

where in each case the final inequality follows by using the standard relation between the geometric and arithmetic mean of non-negative quantities x_i :

$$\frac{1}{n} \sum_{i=1}^n x_i \geq \left(\prod_{i=1}^n x_i \right)^{1/n}. \quad (2.8)$$

2.2 Four-dimensional asymptotically-AdS black holes

These solutions for rotating black holes with pairwise-equal charges in $\mathcal{N} = 8$ gauged supergravity, which were obtained in [27], have the metric

$$\begin{aligned} ds_4^2 &= -\frac{\Delta_r}{W} \left(dt - a \sin^2 \theta \frac{d\phi}{\Xi} \right)^2 + W \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) + \frac{\Delta_\theta \sin^2 \theta}{W} \left[a dt - (r_1 r_2 + a^2) \frac{d\phi}{\Xi} \right]^2, \\ \Delta_r &= r^2 + a^2 - 2m r + g^2 r_1 r_2 (r_1 r_2 + a^2), \\ \Delta_\theta &= 1 - g^2 a^2 \cos^2 \theta, \quad W = r_1 r_2 + a^2 \cos^2 \theta, \\ r_1 &= r + q_1, \quad r_2 = r + q_2, \quad \Xi = 1 - a^2 g^2. \end{aligned} \quad (2.9)$$

The physical charges are proportional to $\sqrt{q_i(q_i + 2m)}$, and reality of the solution implies that the charge parameters q_1 and q_2 should be taken to be non-negative. Special cases include the Kerr-Newman-AdS metric if $q_1 = q_2$, and the Kerr-AdS metric if $q_1 = q_2 = 0$.

The maximum value of $g_{\phi\phi}$ on the horizon is attained at $\theta = \frac{1}{2}\pi$, and it is easily seen that

$$\beta(g) \leq \beta(g; \theta) = \frac{2\pi [(r_+ + q_1)(r_+ + q_2) + a^2]}{\Xi (r_+ + q_1)^{1/2} (r_+ + q_2)^{1/2}}, \quad (2.10)$$

where r_+ , the radius of the horizon, is the largest root of $\Delta_r(r) = 0$.

The mass of the black hole is given by [28]

$$E = \frac{2m + q_1 + q_2}{2\Xi^2}, \quad (2.11)$$

and so conjecture 1 is satisfied if

$$1 + \tilde{q}_1 + \tilde{q}_2 - (1 + \tilde{q}_1)^{1/2} (1 + \tilde{q}_2)^{1/2} + \tilde{a}^2 [1 - (1 + \tilde{q}_1)^{-1/2} (1 + \tilde{q}_2)^{-1/2}] + \tilde{g}^2 [(1 + \tilde{q}_1)(1 + \tilde{q}_2) + \tilde{a}^2] [(1 + \tilde{q}_1)(1 + \tilde{q}_2) + \tilde{a}^2 (1 + \tilde{q}_1)^{-1/2} (1 + \tilde{q}_2)^{-1/2}] \geq 0, \quad (2.12)$$

where we have defined the dimensionless quantities $\tilde{a} = r/r_+$, $\tilde{q}_i = q_i/r_+$ and $\tilde{g} = gr_+$. Clearly the terms in (2.12) proportional to \tilde{g}^2 are always positive, as are the bracketed terms with the \tilde{a}^2 prefactor. The positivity of the remaining terms can be seen easily by squaring:

$$(1 + \tilde{q}_1 + \tilde{q}_2)^2 - (1 + \tilde{q}_1)(1 + \tilde{q}_2) = \tilde{q}_1 + \tilde{q}_2 + \tilde{q}_1^2 + \tilde{q}_2^2 + \tilde{q}_1\tilde{q}_2 \geq 0, \quad (2.13)$$

thus establishing that conjecture 1 is satisfied by these metrics.

2.3 Four-dimensional asymptotically-Melvin black holes

These were first constructed using a Harrison transformation in Einstein-Maxwell theory, by Ernst [29] and in an explicit form by Ernst and Wild [30]. In what follows, we shall see that, perhaps surprisingly, conjecture 1 extends to asymptotically Melvin solutions, at least in the non-rotating case. The metric is

$$ds_4^2 = F^2 \left\{ -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 \right\} + \frac{r^2 \sin^2 \theta}{F^2} d\phi^2, \quad (2.14)$$

with

$$F = 1 + \frac{B^2}{4} r^2 \sin^2 \theta, \quad (2.15)$$

where B is the applied magnetic field. If $m = 0$ we get the Melvin solution, whilst if instead $B = 0$ we get the Schwarzschild solution. The energy with respect to the Melvin background is given simply by [31]

$$E = m, \quad (2.16)$$

and the horizon, which is located at

$$r = 2m, \quad (2.17)$$

has the metric

$$ds^2 = 4m^2 \left\{ (1 + \gamma^2 \sin^2 \theta)^2 d\theta^2 + \frac{\sin^2 \theta}{(1 + \gamma^2 \sin^2 \theta)^2} d\phi^2 \right\}, \quad (2.18)$$

with

$$\gamma = m |B|. \quad (2.19)$$

The area A and temperature T of the horizon are

$$A = 16\pi m^2, \quad T = \frac{1}{8\pi m}. \quad (2.20)$$

Remarkably, these are the same as in the absence of the magnetic field [31].

We have

$$\beta(g) \leq \sup_{\theta} \frac{4\pi E \sin \theta}{1 + \gamma^2 \sin^2 \theta} \quad (2.21)$$

If $\gamma \leq 1$, the circumference $C(\theta)$ has a single maximum at the equator $\theta = \frac{\pi}{2}$, the maximum value being $\frac{4\pi E}{1+\gamma^2} \leq 4\pi E$. If $\gamma \geq 1$, the horizon is dumb-bell shaped and has two maxima with $\gamma \sin \theta = 1$, the maximum value being $\frac{2E\pi}{\gamma} < 4\pi E$. Thus conjecture 1 is always satisfied for this metric. More about the geometry of the horizon may be found in the appendix.

The solutions for a black hole immersed in a magnetic field in Einstein-Maxwell-Dilaton theory have been given by Yazadjiev [32]. He gives results in higher dimensions also, but here we quote the result just for $D = 4$. The main change is that F in (2.14) is replaced by $F^{\frac{1}{1+\alpha^2}}$ where α is the dilaton coupling constant. The area, surface gravity and mass of the solution are independent of α , as is the location of the horizon. The horizon metric is

$$ds^2 = 4E^2 \left\{ (1 + \gamma^2 \sin^2 \theta)^{\frac{2}{1+\alpha^2}} d\theta^2 + \frac{\sin^2 \theta}{(1 + \gamma^2 \sin^2 \theta)^{\frac{2}{1+\alpha^2}}} d\phi^2 \right\}, \quad (2.22)$$

The circumference has a single maximum at $\theta = \frac{\pi}{2}$ as long as $\gamma^2 \leq \frac{1+\alpha^2}{1-\alpha^2}$, otherwise it has two maxima when $\sin^2 \theta = \frac{1+\alpha^2}{\gamma(1-\alpha^2)}$. In all cases $\beta \leq 4\pi E$.

3 Birkhoff Bound in Five Spacetime Dimensions

In this section, we consider $S^1 \times S^1$ sweepouts. In this case these coincide with both the $S^1 \times S^{D-4}$ and the $T^{[\frac{D-1}{2}]}$ sweepouts that we discussed in the introduction. One could instead consider sweepouts by S^2 , but it seems that the more useful is by 2-tori. We shall check conjecture 3, given in (1.8), for two classes of five-dimensional black holes. The first class consists of rotating 3-charge solutions of maximal ungauged supergravity, and the second class consists of charged rotating black holes in minimal gauged supergravity.

For a spherical 3-surface $S = \{S^3, g\}$ and a foliation $f : S \rightarrow \mathbb{R}$ with generic level sets $f^{-1}(c)$ being tori of area $A(c)$ and two singular leaves being linked circles, we define⁵

$$\beta(g; f) = \max_c A(c), \quad (3.1)$$

and

$$\beta(g) = \inf_f \beta(g; f). \quad (3.2)$$

⁵A standard example of such a foliation is to write the round S^3 metric as $ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2$.

3.1 Five-dimensional asymptotically-flat black holes

A convenient expression for the rotating 3-charge asymptotically-flat solutions found in [33] is given in [34]:

$$\begin{aligned}
ds_5^2 &= (H_1 H_2 H_3)^{1/3} (x + y) \left(-\Phi (dt + \mathcal{A})^2 + ds_4^2 \right), \\
ds_4^2 &= \left(\frac{dx^2}{4X} + \frac{dy^2}{4Y} \right) + \frac{U}{G} \left(d\chi - \frac{Z}{U} d\sigma \right)^2 + \frac{XY}{U} d\sigma^2, \\
H_i &= 1 + \frac{2ms_i^2}{x + y}, \quad \Phi = \frac{G}{(x + y)^3 H_1 H_2 H_3}, \\
X &= (x + a^2)(x + b^2) - 2mx, \quad Y = -(a^2 - y)(b^2 - y), \\
G &= (x + y)(x + y - 2m), \quad U = yX - xY, \quad Z = ab(X + Y), \\
\mathcal{A} &= \frac{2mc_1 c_2 c_3}{G} [(a^2 + b^2 - y)d\sigma - abd\chi] - \frac{2ms_1 s_2 s_3}{x + y} (abd\sigma - yd\chi).
\end{aligned} \tag{3.3}$$

The coordinates σ and χ are related to standard azimuthal angles ϕ and ψ with 2π periods by

$$\sigma = \frac{a\phi - b\psi}{a^2 - b^2}, \quad \chi = \frac{b\phi - a\psi}{a^2 - b^2}. \tag{3.5}$$

The x and y coordinates are related to standard radial and latitude coordinates by

$$x = r^2, \quad y = a^2 \cos^2 \theta + b^2 \sin^2 \theta. \tag{3.6}$$

It is straightforward to see that the area of an $S^1 \times S^1$ sweepout on the horizon at a fixed value of θ is given by

$$A(\theta) = \frac{(r^2 c_1 c_2 c_3 + abs_1 s_2 s_3)(r^2 + a^2)(r^2 + b^2) \sin \theta \cos \theta}{r^3 \rho (H_1 H_2 H_3)^{1/6}}, \tag{3.7}$$

evaluated at $r = r_+$, the largest root of $X(r^2) = 0$, where

$$\rho^2 = x + y = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta. \tag{3.8}$$

Unlike the situation in four dimensions, where the S^1 sweepout has its greatest length at the midpoint of the range of the latitude coordinate, here the maximum value $\beta(g; \theta)$ of the $S^1 \times S^1$ sweepout area $A(\theta)$ occurs at a value of θ that is a quite complicated function of the parameters of the solution. Accordingly, in order to test conjecture 3 in this case we shall work with an appropriate upper bound on the sweepout area $A(\theta)$. In order to do this, it is convenient to assume, without loss of generality, that the rotation parameters a and b are ordered such that

$$a^2 \geq b^2. \tag{3.9}$$

Since

$$\rho^2 \geq r^2 + b^2, \quad \sin \theta \cos \theta \leq \frac{1}{2}, \tag{3.10}$$

we shall have

$$\beta(g; \theta) \leq \max_{\theta} A(\theta) \leq \frac{2\pi^2 (r_+^2 c_1 c_2 c_3 + abs_1 s_2 s_3) (r_+^2 + a^2) (r_+^2 + b^2)^{1/2}}{r_+^2 \prod_i (r_+^2 c_i^2 + a^2 s_i^2)^{1/6}}. \tag{3.11}$$

The mass of these black hole solutions is given by [26]

$$E = \frac{1}{4}m\pi \sum_i (c_i^2 + s_i^2), \quad (3.12)$$

Conjecture 3, given in (1.8), is therefore satisfied if

$$\sum_i (c_i^2 + s_i^2) - \frac{3 \left(\prod_i c_i + \tilde{a}\tilde{b} \prod_i s_i \right)}{(1 + \tilde{b}^2)^{1/2} \prod_j (c_j^2 + \tilde{a}^2 s_j^2)^{1/6}} \geq 0, \quad (3.13)$$

where we have defined the dimensionless parameters $\tilde{a} = a/r_+$ and $\tilde{b} = b/r_+$. We can assume that $\tilde{a}\tilde{b}$ is positive, since if it were negative the inequality would be more easily satisfied.

Clearly we have the inequalities

$$\sum_i c_i^2 - \frac{3 \prod_i c_i}{(1 + \tilde{b}^2)^{1/2} \prod_j (c_j^2 + \tilde{a}^2 s_j^2)^{1/6}} \geq \sum_i c_i^2 - \frac{3 \prod_i c_i}{(\prod_j c_j^2)^{1/6}} = \sum_i c_i^2 - 3 \left(\prod_i c_i \right)^{2/3} \geq 0 \quad (3.14)$$

and

$$\begin{aligned} \sum_i s_i^2 - \frac{3\tilde{a}\tilde{b} \prod_i s_i}{(1 + \tilde{b}^2)^{1/2} \prod_j (c_j^2 + \tilde{a}^2 s_j^2)^{1/6}} &\geq \sum_i s_i^2 - \frac{3\tilde{a}\tilde{b} \prod_i s_i}{(1 + \tilde{b}^2)^{1/2} \prod_j (\tilde{a}^2 s_j^2)^{1/6}} \\ &= \sum_i s_i^2 - \frac{3\tilde{b} \left(\prod_i s_i \right)^{2/3}}{(1 + \tilde{b}^2)^{1/2}} \geq 0, \end{aligned} \quad (3.15)$$

where in each case we have used (2.8) in the final step (together with $\tilde{b}/(1 + \tilde{b}^2)^{1/2} \leq 1$ in the second case). Thus we see that the inequality (3.13) holds, and so the five-dimensional 3-charge rotating black holes are indeed consistent with conjecture 3.

3.2 Five-dimensional asymptotically-AdS black holes

The metric and gauge potential for this solution of minimal gauged supergravity are given by [35]

$$\begin{aligned} ds^2 &= -\frac{\Delta_\theta [(1 + g^2 r^2)\rho^2 dt + 2q\nu] dt}{\Xi_a \Xi_b \rho^2} + \frac{2q\nu\omega}{\rho^2} + \frac{f}{\rho^4} \left(\frac{\Delta_\theta dt}{\Xi_a \Xi_b} - \omega \right)^2 + \frac{\rho^2 dr^2}{\Delta_r} \\ &\quad + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{r^2 + a^2}{\Xi_a} \sin^2 \theta d\phi^2 + \frac{r^2 + b^2}{\Xi_b} \cos^2 \theta d\psi^2, \end{aligned} \quad (3.16)$$

$$A = \frac{\sqrt{3}q}{\rho^2} \left(\frac{\Delta_\theta dt}{\Xi_a \Xi_b} - \omega \right), \quad (3.17)$$

where

$$\begin{aligned} \nu &= b \sin^2 \theta d\phi + a \cos^2 \theta d\psi, & \omega &= a \sin^2 \theta \frac{d\phi}{\Xi_a} + b \cos^2 \theta \frac{d\psi}{\Xi_b}, \\ \Delta_\theta &= 1 - a^2 g^2 \cos^2 \theta - b^2 g^2 \sin^2 \theta, & f &= 2m\rho^2 - q^2 + 2abqg^2 \rho^2, \\ \Delta_r &= \frac{(r^2 + a^2)(r^2 + b^2)(1 + g^2 r^2) + q^2 + 2abq}{r^2} - 2m, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, & \Xi_a &= 1 - a^2 g^2, \quad \Xi_b = 1 - b^2 g^2, \end{aligned} \quad (3.18)$$

The determinant of the two-dimensional sub-metric spanned by $d\phi$ and $d\psi$ is given on the horizon at $r = r_+$ by

$$\sqrt{\det(Z_{ij})} = \frac{\sqrt{\Delta_\theta} \sin \theta \cos \theta}{\Xi_a \Xi_b r_+ \rho_+} [(r_+^2 + a^2)(r_+^2 + b^2) + abq], \quad (3.19)$$

where r_+ is the largest root of $\Delta_r(r) = 0$, and ρ_+ means ρ evaluated with $r = r_+$. It is convenient to assume, without loss of generality, that $a^2 \geq b^2$, and so although it is not easy to give the exact expression for $\sqrt{\det(Z_{ij})}$ maximized over θ , we may use the inequalities

$$\rho^2 \geq r^2 + b^2, \quad \Delta_\theta \leq 1, \quad \sin \theta \cos \theta \leq \frac{1}{2} \quad (3.20)$$

in order to obtain the upper bound for the area $A(\theta)$ of the $S^1 \times S^1$ sweepout:

$$\beta(g; \theta) \leq \max A(\theta) \leq \frac{2\pi^2 [(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{\Xi_a \Xi_b r_+ (r_+^2 + b^2)^{1/2}}. \quad (3.21)$$

(Sharper bounds can, of course, be obtained, but this one turns out to suffice.)

The energy of the rotating charged black hole is given by [35]

$$E = \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a \Xi_b) + 2\pi q ab g^2 (\Xi_a + \Xi_b)}{4\Xi_a^2 \Xi_b^2}. \quad (3.22)$$

It is convenient to parametrize the metric by a , b , q and r_+ , with m solved for in terms of these, and the gauge coupling g , by using $\Delta(r_+) = 0$. If we then form the dimensionless quantities

$$\tilde{a} = \frac{a}{r_+}, \quad \tilde{b} = \frac{b}{r_+}, \quad \tilde{q} = \frac{q}{r_+^2}, \quad \tilde{g} = gr_+, \quad (3.23)$$

then using (3.21) and (3.22), conjecture 3 will be verified for this rotating charged black hole if

$$(1 + \tilde{a}^2)(1 + \tilde{b}^2)(1 + \tilde{g}^2) + 2\tilde{a}\tilde{b}\tilde{q} + \frac{8}{3}\tilde{a}\tilde{b}\tilde{g}^2\tilde{q} + \tilde{q}^2 - \left[(1 + \tilde{a}^2)(1 + \tilde{b}^2)^{1/2} + \frac{\tilde{a}\tilde{b}\tilde{q}}{(1 + \tilde{b}^2)^{1/2}} \right] \geq 0. \quad (3.24)$$

It is very easy to see that (3.24) is satisfied if q is assumed to be non-negative. However, in the parametrization used here q can take either sign. We therefore proceed by completing the square on the terms involving \tilde{q} in (3.24). Dropping the positive term $(\tilde{q} + \dots)^2$ implies that (3.24) will be satisfied if the inequality

$$(1 + \tilde{a}^2 + \tilde{b}^2) - \frac{\tilde{a}^2 \tilde{b}^2 (1 - \tilde{b}^2 \tilde{g}^2)}{4(1 + \tilde{b}^2)} - \frac{1}{9} \tilde{a}^2 \tilde{b}^2 \tilde{g}^2 (15 + 16\tilde{g}^2) - \frac{1 + \tilde{a}^2 + \tilde{b}^2 - \frac{4}{3} \tilde{a}^2 \tilde{b}^2 \tilde{g}^2}{(1 + \tilde{b}^2)^{1/2}} \geq 0 \quad (3.25)$$

holds. We have already assumed that $a^2 \geq b^2$, and we must also restrict the rotation parameters such that $\Xi_a > 0$, $\Xi_b > 0$, so we must require that $\tilde{a}^2 \tilde{g}^2 < 1$. A convenient reparametrization that takes account of these conditions and that eliminates the square root in (3.25) is to write

$$\tilde{a} = \frac{1}{2}(d_1 - d_1^{-1}), \quad \tilde{b} = \frac{1}{2}(d_2 - d_2^{-1}), \quad \tilde{g} = \tilde{a}^{-1}(z + 1)^{-1}, \quad (3.26)$$

and then set

$$d_1 = x + y + 1, \quad d_2 = y + 1. \quad (3.27)$$

The parameter space is then spanned by x , y and z lying in the positive octant of \mathbb{R}^3 .

Substituting these definitions into (3.25) then shows that conjecture 3 is satisfied if a certain multinomial $P(x, y, z)$ is positive for all positive x , y and z . $P(x, y, z)$ has 441 terms, of which 438 form a multinomial $Q(x, y, z)$ whose coefficients are all strictly positive, plus 3 remaining terms with negative coefficients:

$$P(x, y, z) = Q(x, y, z) - 324y^{11} - 145y^{12} - 6y^{13}. \quad (3.28)$$

In fact

$$\begin{aligned} P(0, y, 0) = & (1 + y)^2 (2304 + 13824y + 37376y^2 + 60160y^3 + 64320y^4 + 48384y^5 \\ & + 26144y^6 + 9696y^7 + 1968y^8 - 64y^9 - 124y^{10} - 12y^{11} + 3y^{12}), \end{aligned} \quad (3.29)$$

and since $P(x, y, z) - P(0, y, 0)$ has strictly positive coefficients, conjecture 3 will be established if we can show that the dodecadic factor in (3.29) is positive for all positive y . This can be shown by means of a straightforward application of Sturm's sign-sequence theorem [36]. This completes the demonstration that the charged rotating black hole in five-dimensional minimal gauged supergravity satisfies conjecture 3 for $S^1 \times S^1$ sweepouts.

4 Closed Geodesic Bounds

In this section, our aim is to test conjectures 4 and 5 for Kerr-AdS black holes in arbitrary dimensions. In order to do so, we need a bound on the length $\ell(g)$ of the shortest closed non-trivial geodesic. By a theorem of Lyusternik and Fet, every compact Riemannian manifold admits at least one nontrivial closed geodesic [37]. Moreover, it is a long-standing conjecture that there exist infinitely many nontrivial closed geodesics on every compact Riemannian manifold [38]. For any metric on the 3-sphere, it has been shown that there are at least two geometrically distinct closed geodesics [39].

From the results quoted above, we may assume that the horizons discussed in this paper admit a closed geodesic of shortest length. In fact, we may exhibit explicitly at least $\frac{1}{4}(D-1)^2$ closed geodesics in the odd-dimensional Kerr-AdS metrics, and $\frac{1}{4}(D^2-4)$ in the even-dimensional cases.

In four spacetime dimensions, Pu's theorem [20] gives an upper bound for the length of the shortest closed geodesic in terms of the area of the horizon, provided that the horizon admits an antipodal map. All of the horizons we consider in this paper do in fact admit an antipodal symmetry, but it appears that there is no higher-dimensional generalisation of Pu's theorem. Nevertheless, it is still possible to give an upper bound to the length of the shortest closed geodesic, by bounding the length of any curve joining a point and its antipode. This will provide a bound on what is called the "systole," which is defined as the least length of any homotopically nontrivial curve on the quotient of the horizon by the antipodal map [40]. In fact, in what follows we shall estimate the systole by finding closed geodesics that pass through pairs of antipodal points.

4.1 Asymptotically-AdS black holes in higher dimensions

The general Kerr-AdS metrics in arbitrary dimension D were obtained in [41, 42]. They have $N \equiv [(D-1)/2]$ independent rotation parameters a_i in N orthogonal 2-planes. We have $D = 2N + 1$

when D is odd, and $D = 2N + 2$ when D is even. Defining $\epsilon \equiv (D - 1) \bmod 2$, so that $D = 2N + 1 + \epsilon$, the metrics can be described by introducing N azimuthal angles ϕ_i , and $(N + \epsilon)$ “direction cosines” μ_i obeying the constraint

$$\sum_{i=1}^{N+\epsilon} \mu_i^2 = 1. \quad (4.1)$$

In Boyer-Linquist coordinates, the metrics are given by [41, 42]

$$\begin{aligned} ds^2 = & -W(1 + g^2 r^2) dt^2 + \frac{2m}{U} \left(W dt - \sum_{i=1}^N \frac{a_i \mu_i^2 d\phi_i}{\Xi_i} \right)^2 + \sum_{i=1}^N \frac{r^2 + a_i^2}{\Xi_i} \mu_i^2 d\phi_i^2 \\ & + \frac{U dr^2}{V - 2m} + \sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{\Xi_i} d\mu_i^2 - \frac{g^2}{W(1 + g^2 r^2)} \left(\sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{\Xi_i} \mu_i d\mu_i \right)^2, \end{aligned} \quad (4.2)$$

where

$$W \equiv \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{\Xi_i}, \quad U \equiv r^\epsilon \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^N (r^2 + a_j^2), \quad (4.3)$$

$$V \equiv r^{\epsilon-2} (1 + g^2 r^2) \prod_{i=1}^N (r^2 + a_i^2), \quad \Xi_i \equiv 1 - g^2 a_i^2. \quad (4.4)$$

They satisfy $R_{\mu\nu} = -(D - 1) g^2 g_{\mu\nu}$. The horizon is located at $r = r_+$, where r_+ is the largest root of $V(r) = 2m$. The induced metric on the horizon is obtained by setting $r = r_+$ and $t = \text{constant}$ in (4.2).

For our purposes it will prove more illuminating to introduce $2N$ Cartesian coordinates (x_i, y_i) and, in even spacetime dimensions, where $\epsilon = 1$, an additional coordinate z , such that

$$x_i + iy_i = \mu_i e^{i\phi_i}, \quad z = \epsilon \mu_{N+\epsilon}. \quad (4.5)$$

The constraint (4.1) becomes

$$\sum_{i=1}^N (x_i^2 + y_i^2) + z^2 = 1, \quad (4.6)$$

which defines a round hypersphere in $\mathbb{E}^{2N+\epsilon}$. One has, for each i ,

$$\begin{aligned} \mu_i d\mu_i &= x_i dx_i + y_i dy_i, & \epsilon \mu_{N+\epsilon} d\mu_{N+\epsilon} &= z dz, \\ \mu_i^2 d\phi_i &= x_i dy_i - y_i dx_i, \\ d\mu_i^2 + \mu_i^2 d\phi_i^2 &= dx_i^2 + dy_i^2. \end{aligned} \quad (4.7)$$

On the horizon, we have the following commuting \mathbb{Z}_2 isometries:

$$\begin{aligned} A : & \quad x_i \longrightarrow -x_i, \\ B : & \quad y_i \longrightarrow -y_i, \\ C_i : & \quad (x_i, y_i) \longrightarrow (-x_i, -y_i) \quad \text{for each } i, \\ D : & \quad z \longrightarrow -z. \end{aligned} \quad (4.8)$$

In each case, those coordinates that are not specified are left unchanged by the map. For the maps A and B , all N of the x_i or y_i coordinates undergo a sign reversal. For the map C_i , only the x_i and y_i coordinates for the specified value of i undergo a sign reversal. The product ABD is the antipodal map

$$(x_i, y_i, z) \rightarrow (-x_i, -y_i, -z). \quad (4.9)$$

The fixed-point sets of any product of A , B , C_i , D are totally-geodesic submanifolds. If the fixed-point set is one-dimensional it is a geodesic; if it is two-dimensional, it is a minimal (strictly, extremal) 2-surface; etc. All of the isometries lift to the whole spacetime provided that either t is unchanged or $t \rightarrow -t$ as appropriate.

Using these facts, one easily shows that the following circles are geodesic:

$$\begin{aligned} D = 2N + 2 : \quad & x_i^2 + z^2 = 1, & 1 \leq i \leq N, \\ & y_i^2 + z^2 = 1, & 1 \leq i \leq N, \\ & x_i^2 + x_j^2 = 1, & 1 \leq i < j \leq N, \\ & y_i^2 + y_j^2 = 1, & 1 \leq i < j \leq N, \\ & x_i^2 + y_i^2 = 1, & 1 \leq i \leq N, \\ \\ D = 2N + 1 : \quad & x_i^2 + x_j^2 = 1, & 1 \leq i < j \leq N, \\ & y_i^2 + y_j^2 = 1, & 1 \leq i < j \leq N, \\ & x_i^2 + y_i^2 = 1, & 1 \leq i \leq N. \end{aligned} \quad (4.10)$$

Acting with the isometry group T^N , which corresponds to rotations in each of the (x_i, y_i) planes, one obtains continuous families of such circular geodesics. Thus, for generic values of the a_i rotation parameters, there are $\frac{1}{2}N(N-1) + 2N = \frac{1}{8}(D-2)(D+4)$ classes of geometrically distinct closed geodesics in even dimensions, and $\frac{1}{2}N(N-1) + N = \frac{1}{8}(D^2-1)$ in odd dimensions. In what follows, we shall select the closed geodesics that give the optimal estimate for $\ell(g)$.

4.1.1 $D = 2N + 2$ dimensions

Let us assume, without loss of generality, that the rotation parameters are ordered so that

$$a_1^2 \leq a_2^2 \leq a_3^2 \cdots \leq a_N^2. \quad (4.11)$$

Points on the horizon of the form $x_1^2 + z^2 = 1$, with all other x_i and all y_i vanishing, are invariant under the product $B \prod_{i \geq 2} C_i$ of the \mathbb{Z}_2 isometries defined in (4.8). The curve defined by $x_1^2 + z^2 = 1$, which may be parameterised by

$$x_1 = \sin \psi, \quad z = \cos \psi, \quad (4.12)$$

for $0 \leq \psi \leq 2\pi$, is therefore a closed geodesic. Points ψ and $\psi + \pi$ on the curve are antipodal.

We see from (4.2) that the length of this geodesic, $L = \int ds$, is bounded from above by taking

$$ds^2 \leq \left[r_+^2 \cos^2 \psi + \frac{r_+^2 + a_1^2}{\Xi_1} \sin^2 \psi \right] d\psi^2, \quad (4.13)$$

with equality if $g = 0$. Clearly we may then obtain the bound

$$ds^2 \leq \frac{r_+^2 + a_1^2}{\Xi_1} d\psi^2, \quad (4.14)$$

and hence, in view of (4.11), the length L of the shortest closed geodesic of this type is bounded by

$$L \leq \frac{2\pi (r_+^2 + a_1^2)^{1/2}}{\Xi_1^{1/2}}. \quad (4.15)$$

(Note that equality holds in the Schwarzschild limit.)

The area of the horizon in $D = 2N + 2$ dimensions is given by

$$A = \mathcal{A}_{D-2} \prod_i \frac{r_+^2 + a_i^2}{\Xi_i}, \quad (4.16)$$

where

$$\mathcal{A}_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma[(D-1)/2]} \quad (4.17)$$

is the volume of the unit $(D-2)$ -sphere. It then follows that

$$\left(\frac{l}{2\pi}\right)^{D-2} \leq \left(\frac{A}{\mathcal{A}_{D-2}}\right). \quad (4.18)$$

Thus conjecture 5, given in (1.11), is obeyed in this case.

The energy of the Kerr-AdS metric in $D = 2N + 2$ dimensions is given by [43]

$$E = \frac{m \mathcal{A}_{D-2}}{4\pi \prod_i \Xi_i} \sum_{j=1}^N \frac{1}{\Xi_j}. \quad (4.19)$$

For the Schwarzschild limit we have $E = mN \mathcal{A}_{D-2}/(4\pi)$ and $m = \frac{1}{2}r_+^{2N-1}$. Conjecture 4, given in (1.9), is thus

$$\frac{8\pi E}{N \mathcal{A}_{D-2}} - \left(\frac{L}{2\pi}\right)^{2N-1} \geq 0, \quad (4.20)$$

which is saturated in the Schwarzschild limit.

To test conjecture 4 for the Kerr-AdS metric in $D = 2N + 2$ dimensions, we may use the inequality (4.15), and check to see whether

$$\frac{(1 + g^2 r_+^2)}{N r_+} \prod_{i=1}^N \frac{r_+^2 + a_i^2}{\Xi_i} \sum_{j=1}^N \frac{1}{\Xi_j} - \frac{(r_+^2 + a_1^2)^{N-\frac{1}{2}}}{\Xi_1^{N-\frac{1}{2}}} \geq 0. \quad (4.21)$$

Reorganizing this as

$$(1 + g^2 r_+^2) \left(1 + \frac{a_1^2}{r_+^2}\right)^{1/2} \frac{\prod_{i=1}^N (r_+^2 + a_i^2)}{(r_+^2 + a_1^2)^N} \left(\prod_{j=1}^N \frac{\Xi_1}{\Xi_j}\right) \frac{1}{\sqrt{\Xi_1}} \left(\frac{1}{N} \sum_{k=1}^N \frac{1}{\Xi_k}\right) - 1 \geq 0, \quad (4.22)$$

we observe that in view of (4.11), every factor in the first term is greater than or equal to 1, and hence conjecture 4 is indeed satisfied by Kerr-AdS in all even dimensions.

4.1.2 $D = 2N + 1$ dimensions

The case of odd spacetime dimensions is very similar. The curve $x_1^2 + x_2^2 = 1$, with all other x_i and all y_i vanishing, is easily seen, by arguments similar to those above, to be a closed geodesic. If

$$x_1 = \sin \psi, \quad x_2 = \cos \psi, \quad (4.23)$$

with $0 \leq \psi \leq 2\pi$, then again points ψ and $\psi + \pi$ are antipodal. The length of the closed curve is bounded above by

$$L \leq \int_0^{2\pi} \left(\frac{r_+^2 + a_1^2}{\Xi_1} \sin^2 \psi + \frac{r_+^2 + a_2^2}{\Xi_2} \cos^2 \psi \right)^{1/2} d\psi, \quad (4.24)$$

with equality when $g = 0$. If we again assume the rotation parameters are ordered as in (4.11), we obtain the bound

$$L \leq \frac{2\pi(r_+^2 + a_2^2)^{1/2}}{\Xi_2^{1/2}}. \quad (4.25)$$

The area of the horizon in the case of odd spacetime dimensions is given by

$$A = \frac{\mathcal{A}_{D-2}}{r_+} \prod_i \frac{r_+^2 + a_i^2}{\Xi_i}. \quad (4.26)$$

We find that the bound (4.25) is too weak to provide support for conjecture 5 in this odd-dimensional case.

The mass of the odd-dimensional Kerr-AdS black hole is given by [43]

$$E = \frac{m \mathcal{A}_{D-2}}{4\pi \prod_i \Xi_i} \left(\sum_j \frac{1}{\Xi_j} - \frac{1}{2} \right). \quad (4.27)$$

The inequality in conjecture 4, which is saturated in the Schwarzschild limit, is then given by

$$\frac{16\pi E}{(2N-1) \mathcal{A}_{D-2}} - \left(\frac{L_{\min}}{2\pi} \right)^{2N-2} \geq 0. \quad (4.28)$$

Substituting the results obtained above, we therefore find that conjecture 4 will be satisfied if

$$\frac{(1 + g^2 r_+^2)}{\Xi_1} \frac{(r_+^2 + a_1^2)}{r_+^2} \frac{1}{(2N-1)} \left(2 \sum_j \frac{1}{\Xi_j} - 1 \right) \prod_{k=3}^N \left(\frac{\Xi_2}{\Xi_k} \times \frac{r_+^2 + a_k^2}{r_+^2 + a_2^2} \right) - 1 \geq 0. \quad (4.29)$$

In view of (4.11), we see that conjecture 4 is indeed satisfied by Kerr-AdS black holes in all odd dimensions.

5 Sweepouts by Higher-Dimensional Spheres

In this section we shall consider more general sweepouts by products of spheres $S^p \times S^q$ where $p + q = D - 3$. The definitions of $\beta(g; f)$ and $\beta(g)$ are completely analogous to those given earlier.

5.1 $S^1 \times S^{D-4}$ sweepouts in $D = 2N + 1$ Kerr-AdS

This can be applied conveniently in the case that all the rotation parameters are equal. The metric in $D = 2N + 1$ dimensions is then given by [43]

$$ds^2 = -\frac{(1 + g^2 r^2) dt^2}{\Xi} + \frac{U dr^2}{V - 2m} + \frac{r^2 + a^2}{\Xi} [(d\psi + A)^2 + d\Sigma_{N-1}^2] + \frac{2m}{U\Xi^2} [dt - a(d\psi + A)]^2, \quad (5.1)$$

where

$$U = (r^2 + a^2)^{N-1}, \quad V = \frac{1}{r^2} (r^2 + a^2)^N (1 + g^2 r^2), \quad \Xi = 1 - g^2 a^2, \quad (5.2)$$

and A is a potential for the Kähler form of the Fubini-Study metric $d\Sigma_{N-1}^2$ on CP^{N-1} . We may write the metric on CP^{N-1} in terms of the Fubini-Study metric on CP^{N-2} as [44]

$$d\Sigma_{N-1}^2 = d\xi^2 + \sin^2 \xi \cos^2 \xi (d\tau + B)^2 + \sin^2 \xi d\Sigma_{N-2}^2, \quad (5.3)$$

where B is a potential for the Kähler form of $d\Sigma_{N-2}^2$. The level surfaces of (5.3) at constant ξ are squashed $(2N - 3)$ -spheres, degenerating to a point at $\xi = 0$ and to a CP^{N-2} bolt at $\xi = \frac{1}{2}\pi$. The volume of the $(2N - 3)$ -sphere at a given ξ is

$$\begin{aligned} V_{2N-3} &= 2\pi \sin^{N-1} \xi \cos \xi \Sigma_{N-2}, \\ &= \sin^{N-1} \xi \cos \xi \Sigma_{N-2} \mathcal{A}_{2N-3}, \end{aligned} \quad (5.4)$$

where Σ_{N-2} is the volume of the “unit” CP^{N-2} metric, and hence $2\pi \Sigma_{N-2} = \mathcal{A}_{2N-3}$, the volume of the unit round $(2N - 3)$ -sphere. V_{2N-3} attains its maximum volume at $\cos \xi = 1/\sqrt{N}$, and hence

$$V_{2N-3}^{\max} = (N - 1)^{\frac{1}{2}(N-1)} N^{-\frac{1}{2}N} \mathcal{A}_{2N-3}. \quad (5.5)$$

We may now determine the $S^1 \times S^{2N-3}$ hyperhoop volume,⁶ where the S^1 is parameterized by the Hopf fibre coordinate ψ and the S^{2N-3} is the equatorial sphere obtained above. In the Kerr-AdS metric (5.1) we therefore find

$$V_{\text{hoop}} = \frac{2\pi}{r_+} \left(\frac{r_+^2 + a^2}{\Xi} \right)^{N-\frac{1}{2}} (N - 1)^{\frac{1}{2}(N-1)} N^{-\frac{1}{2}N} \mathcal{A}_{2N-3}. \quad (5.6)$$

The energy of the Kerr-AdS metric is given by [43]

$$\begin{aligned} E &= \frac{m(2N - \Xi) \mathcal{A}_{2N-1}}{8\pi \Xi^{N+1}}, \\ &= \frac{(2N - \Xi)(r_+^2 + a^2)^N (1 + g^2 r_+^2) \mathcal{A}_{2N-1}}{16\pi \Xi^{N+1}}. \end{aligned} \quad (5.7)$$

The hyperhoop inequality of conjecture 6 for black holes of dimension $D = 2N + 1$ is given in (1.13). Verifying this for Kerr-AdS requires showing that

$$\left(1 + \frac{a^2}{r_+^2}\right)^{1/2} (1 + g^2 r_+^2) \frac{1}{\sqrt{\Xi}} \left(\frac{2N - \Xi}{2N - 1}\right) - 1 \geq 0. \quad (5.8)$$

Each of the factors in the first term is manifestly greater than or equal to 1, and hence conjecture 6 holds.

⁶In dimensions higher than five, we shall always refer to volumes, rather than lengths or areas as we did in four and five dimensions, when describing codimension-one hyperhoops.

5.2 Sweepouts of S^{D-2} by $S^p \times S^q$

There are numerous ways of sweeping out S^{D-2} . Among them are sweepouts by products of spheres. Thus on the unit round $(D-2)$ -sphere, with $D-2 = (p+q+1)$, we may write the metric as

$$d\Omega_{p+q+1}^2 = d\theta^2 + \sin^2 \theta d\Omega_p^2 + \cos^2 \theta d\Omega_q^2. \quad (5.9)$$

The $(p+q)$ -volume of the $S^p \times S^q$ hyperhoop,

$$V_{p,q}(\theta) = \sin^p \theta \cos^q \theta \mathcal{A}_p \mathcal{A}_q, \quad (5.10)$$

is maximized at $\sin \theta = \sqrt{\frac{p}{p+q}}$, $\cos \theta = \sqrt{\frac{q}{p+q}}$. The maximum value of $V_{p,q}$ is therefore

$$V_{p,q} = \left(\frac{p}{p+q}\right)^{\frac{p}{2}} \left(\frac{q}{p+q}\right)^{\frac{q}{2}} \frac{4\pi^{\frac{p+q+2}{2}}}{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}. \quad (5.11)$$

In this way we obtain different upper bounds for the Birkhoff invariant, depending on which sweepout we use. Presumably the actual value of the Birkhoff invariant will also depend on how we sweep out the sphere. It sometimes happens, because of a suitable symmetry group, that one may define sweepouts by products of spheres even if the metric on the horizon is not the round one. In what follows we shall do this for static black holes immersed in a magnetic field and to rotating black holes with a single nonvanishing angular velocity.

5.2.1 Higher-dimensional magnetic fields

The metric of a Tangherlini black hole immersed in a magnetic field in D spacetime dimensions has been given by Ortaggio [45]. The horizon metric is

$$ds_H^2 = F_+^{\frac{2}{D-3}} r_+^2 [\cos^2 \theta d\Omega_{D-4}^2 + d\theta^2] + F_+^{-2} r_+^2 \sin^2 \theta d\phi^2, \quad (5.12)$$

with $F_+ \equiv F(r=r_+)$,

$$F = 1 + \frac{(D-3)}{2(D-2)} B^2 \rho^2, \quad (5.13)$$

$\rho = r \sin \theta$, and $d\Omega_{D-4}^2$ is the standard round metric on a unit S^{D-4} .

The level sets $\theta = \text{constant}$ now provide a foliation or sweepout of the horizon, whose nonsingular leaves have topology $S^\times S^{D-3}$. There are two critical level sets, $\theta = 0$ and $\theta = \frac{\pi}{2}$. The first is an S^{D-4} , the second an S^1 . Note that if $D = 5$, the sweepout is by *Clifford tori* $S^1 \times S^1$.

The $(D-3)$ -volume of the hyperhoops with $\theta = \text{constant}$ is

$$V(\theta) = \pi F_+^{-\frac{1}{D-3}} r_+^{D-3} \mathcal{A}_{D-4} \cos^{D-5} \theta \sin 2\theta. \quad (5.14)$$

Clearly

$$V(\theta) \leq \pi r_+^{D-3} \mathcal{A}_{D-4} \cos^{D-5} \theta \sin 2\theta. \quad (5.15)$$

It follows that the magnetic field reduces the maximum value of $V(\theta)$ below the maximum value for $V(\theta)$ on a round sphere for this type of sweepout. This is given by the value of the right-hand side of (5.15) when $\cos \theta = \sqrt{\frac{D-5}{D-4}}$. This establishes an inequality of the same type as conjecture 6 for this type of sweepout. However it remains unclear how this type of sweepout compares with other types of sweepout.

5.2.2 Non-rotating Einstein-Maxwell-Dilaton black holes in higher dimensions

This was dealt with by Yazadjiev [32]. The general results of Ortaggio [45] go through with F in (5.12) replaced by $F^{\frac{1}{1+\alpha^2}}$, where α a dilaton coupling constant. Thus the results of the previous section, for which $\alpha = 0$, still go through.

5.2.3 Kerr-AdS with a single rotation parameter

Geometrically this is very similar to the magnetic field case discussed above. If the magnetic field vanishes, the metric on the horizon is

$$ds^2 = \frac{(1 - a^2 g^2 \cos^2 \theta)}{(r_+^2 + a^2 \cos^2 \theta)} \frac{(r_+^2 + a^2)^2}{(1 - a^2 g^2)^2} \sin^2 \theta d\phi^2 + r_+^2 \cos^2 \theta d\Omega_{D-4}^2 + \frac{(r_+^2 + a^2 \cos^2 \theta)}{(1 - a^2 g^2 \cos^2 \theta)} d\theta^2, \quad (5.16)$$

with $0 \leq \theta \leq \frac{\pi}{2}$. Again we have a foliation by $S^1 \times S^{D-4}$, and

$$V(\theta) = \pi \mathcal{A}_{D-4} r_+^{D-4} \sqrt{\frac{1 - a^2 g^2 \cos^2 \theta}{r_+^2 + a^2 \cos^2 \theta}} \frac{(r_+^2 + a^2)}{(1 - a^2 g^2)} \cos^{D-5} \theta \sin 2\theta. \quad (5.17)$$

The argument now is almost the same as with the magnetic field, and again conjecture 6 holds for this sweepout.

5.2.4 Myers-Perry with an applied magnetic field along one non-rotating direction

As mentioned above, applying a magnetic field to a rotating or charged black hole produces quite complicated results owing to various induction effects. However, if the magnetic field lies as along a direction (i.e. a two-plane direction) about which the black hole is not rotating, then Yazadjiev [32] has shown that even in Einstein-Maxwell dilaton theory the metric remains remarkably simple. In the case of odd spacetime dimension D , with one rotation parameter (vanishing in the 1-2 plane, say) and the Maxwell 2-form's having “legs” only in the 1-2 direction, then the mass, area, location and thermodynamics remain unchanged. If μ_1, ϕ_1 are the relevant coordinates associated with the 1-2 plane, so that $a_1 = 0$, then the horizon metric is given by eqn (68) in [32]:

$$\begin{aligned} ds_H^2 &= F_+^{-\frac{2}{(D-3)(1+\alpha^2)}} \left\{ \sum_{i=2}^{\frac{D-1}{2}} (r_+^2 + a_i^2) (d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{m}{U_+} \left(\sum_{i=2}^{\frac{D-2}{2}} a_i \mu_i d\phi \right)^2 + r_+^2 d\mu_1^2 \right\} \\ &+ F_+^{-\frac{2}{1+\alpha^2}} r_+^2 \mu_1^2 d\phi_1^2, \\ U &= \sum_i \frac{\mu_i^2}{r^2 + a_i^2} \prod_j (r^2 + a_j^2). \end{aligned} \quad (5.18)$$

Now we consider the foliation whose $S^1 \times S^{D-4}$ leaves are given by $\mu_1 = \text{constant}$. The volume $V(\mu_1, B)$ of such leaves (where B is the magnetic field) satisfies

$$V(\mu_1, B) = F_+^{-\frac{1}{(D-3)(1+\alpha^2)}} V(\mu_1, 0) \leq V(\mu_1, 0), \quad (5.19)$$

and so the magnetic field can only have the effect of reducing any upper bound for the Birkhoff invariant. Thus if conjecture 6 is satisfied in the absence of a magnetic field, it will be satisfied in its presence.

6 Conclusions

In this paper we have tested some conjectures [14] relating the geometry of apparent horizons and their total energy in four spacetime dimensions, and generalized then to higher-dimensional spacetimes. We expect their validity to depend on a suitable energy condition, but this is presumably weaker than the dominant energy condition, since the latter does not hold in our gauged supergravity examples. The total energy can be defined in asymptotically-flat, asymptotically-AdS and asymptotically-Melvin spacetimes. So far we have found support for our conjectures in all even dimensions. In odd spacetime dimensions we found support for conjectures 4 and 6, but we were unable to make a statement about conjecture 5 (which relates the ratio of the length of the shortest non-trivial geodesic to the cube root of the area). The absence of support for conjecture 5 in odd dimensions is because our upper bound on the length of the shortest nontrivial closed geodesic is too weak to be decisive. If it is in fact a good estimate for $\ell(g)$, then conjecture 5 would fail in odd dimensions.

The differences between even and odd dimensions are rather striking, and may be related to other differences in the properties of black holes in even and odd dimensions; for example stability and uniqueness. For instance, there is growing evidence that the geometry of the horizon plays an important role in determining stability.

Of course failure to find a contradiction to a conjecture is not a proof, merely “circumstantial evidence.” However in the course of the investigation we found that to establish the necessary inequalities required some far from obvious manipulations. This, together the number of non-trivial examples, gives us some confidence that the conjectures that have held up so far may indeed be true. It may be possible to give some partial proofs for such configurations as collapsing shells, along the lines of what was done in four spacetime dimensions in [14]. On the other hand we invite the skeptical reader to provide counter-examples

7 Acknowledgments

We would like to thank Thiti Sirithanakorn for help during an early part of this investigation. We thank the Mitchell family for their generous hospitality at Cook’s Branch Conservancy during the completion of this work. G.W.G. is grateful for hospitality at the University of Pennsylvania during the early stages of this work. M.C. is supported in part by DOE grant DE-FG05-95ER40893-A020, NSF RTG grant DMS-0636606, the Fay R. and Eugene L. Langberg Chair, and the Slovenian Research Agency (ARRS). The research of C.N.P. is supported in part by DOE grant DE-FG03-95ER40917.

A Isometric embedding of the Ernst-Wild horizon in \mathbb{E}^3

In what follows we give more details of the horizon geometry and correct some statements in [46].

If the surface can be isometrically embedded as a surface of revolution in \mathbb{E}^3 then we must have (in units in which $2E = 1$)

$$\rho = \frac{\sin \theta}{(1 + \gamma^2 \sin^2 \theta)} \quad (\text{A.1})$$

and

$$dz^2 + d\rho^2 = (1 + \gamma^2 \sin^2 \theta)^2 d\theta^2, \quad (\text{A.2})$$

where z , ρ and ϕ are cylindrical coordinates for \mathbb{E}^3 . We have

$$d\rho = \frac{1 - \gamma^2 \sin^2 \theta}{(1 + \gamma^2 \sin^2 \theta)^2} \cos \theta d\theta, \quad (\text{A.3})$$

$$dz = \frac{d\theta}{(1 + \gamma^2 \sin^2 \theta)^2} \sqrt{(1 + \gamma^2 \sin^2 \theta)^6 - \cos^2 \theta (1 - \gamma^2 \sin^2 \theta)^2}. \quad (\text{A.4})$$

Now $(1 + \gamma^2 \sin^2 \theta)^6 \geq (1 - \gamma^2 \sin^2 \theta)^2$ and $\cos^2 \theta \leq 1$, and so the argument of the square root is positive for all θ and for all γ . Thus the embedding is global for all γ , contrary to a statement by Wild and Kerns [46].

If $\gamma < 1$, the surface is convex circumference $C(\theta)$ of the circular leaves of the foliation given by $\theta = \text{constant}$, i.e. by horizontal planes orthogonal to the axis of symmetry is greatest on the equator $\theta = \frac{\pi}{2}$ where it takes the value (restoring units),

$$C\left(\frac{\pi}{2}\right) = \frac{4\pi E}{1 + \gamma^2} \leq 4\pi E. \quad (\text{A.5})$$

However if $\gamma > 1$, then in the interval

$$\arcsin \frac{1}{\gamma} < \theta < \pi - \arcsin \frac{1}{\gamma}, \quad (\text{A.6})$$

which we call the waist region,

$$\frac{d\rho}{dz} < 0 \quad (\text{A.7})$$

which means that the surface becomes dumb-bell shaped and hence non-convex. As a consequence the Gauss curvature becomes negative in a neighbourhood of the equator, as correctly observed in [46]. However that does not preclude a global isometric embedding into Euclidean 3-space, as we have seen.

Recall that Thorne's hoop conjecture was that

Horizons form when and only when a mass E gets compacted into a region whose circumference in EVERY direction is $C \leq 4\pi E$.

The capitalization "EVERY" was intended to emphasise the fact that while for the collapse of oblate shaped bodies, the circumferences are all roughly equal, in the prolate case, the collapse of a long almost cylindrically shaped body whose girth was nevertheless small would not necessarily produce a horizon. However, the polar circumference of the Schwarzschild-Melvin black hole is

$$C_p = 4E \int_0^\pi (1 + \gamma^2 \sin^2 \theta) d\theta = 4\pi E \left(1 + \frac{1}{2}\gamma^2\right) \geq 4\pi E. \quad (\text{A.8})$$

Evidently, the ratio $C_p/(4\pi E)$ may be made arbitrarily large by choosing γ to be arbitrarily large. Thus if one were to interpret C_p as the circumference in directions orthogonal to the axis of symmetry, then Thorne's conjecture would fail.

References

- [1] K.S. Thorne, *Nonspherical gravitational collapse: A short review* in *Magic without Magic* ed. J. Klauder (San Francisco: Freeman) (1972).
- [2] R. Schoen and S.-T. Yau, *The existence of a black hole due to condensation of matter*, Comm. Math. Phys. **90**, 575 (1983).
- [3] K.P. Tod, *The hoop conjecture and the Gibbons-Penrose construction of trapped surfaces*, Class. Quantum Grav. **9**, 1581 (1992).
- [4] J.M.M. Senovilla, *A reformulation of the hoop conjecture*, Europhys. Lett. **81**, 20004 (2008), arXiv:0709.0695 [gr-qc].
- [5] D. Ida, K.i. Nakao, M. Siino and S.A. Hayward, *Hoop conjecture for colliding black holes*, Phys. Rev. **D58**, 121501 (1998).
- [6] M.W. Choptuik and F. Pretorius, *Ultra relativistic particle collisions* Phys. Rev. Lett. **104**, 111101 (2010), arXiv:0908.1780 [gr-qc].
- [7] D.M. Eardley and S.B. Giddings, *Classical black hole production in high-energy collisions*, Phys. Rev. **D66**, 044011 (2002), gr-qc/0201034.
- [8] C. Barrabes, V.P. Frolov and E. Lesigne, *Geometric inequalities and trapped surfaces in higher dimensional spacetimes*, Phys. Rev. **D69**, 101501(2004), gr-qc/0402081.
- [9] Y. Yamada and H.a. Shinkai, *Black objects and hoop conjecture in five-dimensional space-time*, Class. Quant. Grav. **27**, 045012 (2010), arXiv:0907.2570 [gr-qc].
- [10] D. Ida and K.i. Nakao, *Isoperimetric inequality for higher-dimensional black holes*, Phys. Rev. **D66**, 064026 (2002), gr-qc/0204082.
- [11] H. Yoshino and Y. Nambu, *High-energy head-on collisions of particles and hoop conjecture*, Phys. Rev. **D66**, 065004 (2002), gr-qc/0204060.
- [12] C.m.M. Yoo, H. Ishihara, M. Kimura and S. Tanzawa, *Hoop conjecture and the horizon formation cross-section in Kaluza-Klein spacetimes*, Phys. Rev. **D81**, 024020 (2010), arXiv:0906.0689 [gr-qc].
- [13] S. Basu and D. Mattingly, *Asymptotic safety, asymptotic darkness, and the hoop conjecture in the extreme UV*, arXiv:1006.0718.
- [14] G.W. Gibbons, *Birkhoff's invariant and Thorne's hoop conjecture*, arXiv:0903.1580 [gr-qc].
- [15] S.W. Hawking, *Black holes in general relativity*, Comm. Math. Phys. **25**, 152 (1972).
- [16] G.W. Gibbons, *The time symmetric initial value problem for black holes*, Commun. Math. Phys. **27**, 87 (1972).
- [17] G.W. Gibbons, *Some aspects of gravitational radiation and gravitational collapse*, Ph.D. Thesis, University of Cambridge (1972).

- [18] S.W. Hawking, *The event horizon*, in *Black Holes (Les astres occlus)*, C.M. and B. DeWitt (1973), 1-55.
- [19] G.D. Birkhoff, *Dynamical systems with two degrees of freedom*, Trans. Amer. Math. Soc. **18** (1918).
- [20] P.M. Pu, *Some inequalities in certain non-orientable Riemannian manifolds*, Pacific J. Math. **2**, 55 (1952).
- [21] M. Mars, *Present status of the Penrose inequality*, Class. Quant. Grav. **26**, 193001 (2009).
- [22] T. H. Colding and C. De Lellis, *The min-max construction of minimal surfaces*, in *Surveys in Differential Geometry VIII*, math-dg/0303305.
- [23] H.L. Bray and D.A. Lee, *On the Riemannian Penrose inequality in dimensions less than eight*, Duke Mathematical Journal, **148**, 81 (2009).
- [24] C.B. Croke, *Volume and lengths on a three-sphere*, Comm. Anal. Geom. **10**, 467 (2002), available @ <http://www.math.upenn.edu/~ccroke/papers.html>.
- [25] C.B. Croke, *Volume and lengths on a three-sphere*, Proc. Amer. Math. Soc. **136**, 715 (2008), available @ <http://www.math.upenn.edu/~ccroke/papers.html>.
- [26] M. Cvetič and D. Youm, *Entropy of non-extreme charged rotating black holes in string theory*, Phys. Rev. **D54**, 2612 (1996), hep-th/9603147.
- [27] Z.W. Chong, M. Cvetič, H. Lü and C.N. Pope, *Charged rotating black holes in four-dimensional gauged and ungauged supergravities*, Nucl. Phys. **B717**, 246 (2005), hep-th/0411045.
- [28] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Rotating black holes in gauged supergravities: Thermodynamics, supersymmetric limits, topological solitons and time machines*, hep-th/0504080.
- [29] F.J. Ernst, *Black holes in a magnetic universe*, J. Math. Phys. **17**, 54 (1976).
- [30] F.J. Ernst and W. Wild, *Kerr black holes in a magnetic universe*, J. Math. Phys. **17**, 182 (1976).
- [31] E. Radu, *A note on Schwarzschild black hole thermodynamics in a magnetic universe*, gr-qc/0122035.
- [32] S.S. Yazadjiev, *Magnetized black holes and black rings in the higher dimensional dilaton gravity*, Phys. Rev. **D73**, 064008 (2006), gr-qc/0511114.
- [33] M. Cvetič and D. Youm, *General rotating five dimensional black holes of toroidally compactified heterotic string*, Nucl. Phys. **B476**, 118 (1996), hep-th/9603100.
- [34] Z.W. Chong, M. Cvetič, H. Lü and C.N. Pope, *Non-extremal rotating black holes in five-dimensional gauged supergravity*, Phys. Lett. **B644**, 192 (2007), hep-th/0606213.

- [35] Z.W. Chong, M. Cvetič, H. Lü and C.N. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, Phys. Rev. Lett. **95**, 161301 (2005), hep-th/0506029.
- [36] C.F. Sturm, *Résolution des équations algébriques*, Bulletin de Férussac, **11**, 419 (1829).
- [37] L.A. Lyusternik and A.I. Fet, *Variational problems on closed manifolds*, Dokl. Akad. Nauk SSSR (N.S.) **81**, 17 (1951).
- [38] S.T. Yau, Problem section in: S.T. Yau (Ed.), *Seminar on differential geometry*, Princeton Univ. Press, Princeton (1982), 669-706.
- [39] Y. Long and H. Duan, *Multiples closed geodesics on 3-spheres*, Adv. Math. **221**, 1757 (2009).
- [40] M. Berger, *What is a systole?*, Notices of the AMS, **55**, 374 (2008).
- [41] G.W. Gibbons, H. Lü, D.N. Page and C.N. Pope, *The general Kerr-de Sitter metrics in all dimensions*, J. Geom. Phys. **53**, 49 (2005), hep-th/0404008.
- [42] G.W. Gibbons, H. Lü, D.N. Page and C.N. Pope, *Rotating black holes in higher dimensions with a cosmological constant*, Phys. Rev. Lett. **93**, 171102 (2004), hep-th/0409155.
- [43] G.W. Gibbons, M.J. Perry and C.N. Pope, *The first law of thermodynamics for Kerr-anti-de Sitter black holes*, Class. Quantum Grav. **22**, 1503 (2005), hep-th/0408217.
- [44] P. Hoxha, R.R. Martinez-Acosta and C.N. Pope, *Kaluza-Klein consistency, Killing vectors, and Kähler spaces*, Class. Quant. Grav. **17**, 4207 (2000), hep-th/0005172.
- [45] M. Ortogio, *Higher dimensional black holes in external magnetic fields*, gr-qc/0410048.
- [46] W.J. Wild and R.M. Kerns, *Surface geometry of a black hole in a magnetic field*, Phys. Rev. **D21**, 332 (1980).